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LETTER TO THE EDITOR

Quantum phase transitions from a new class of representations of a diffeomorphism group

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Abstract. Using a technique based on self-similar random processes, we construct unitary representations of the group of diffeomorphisms of $\mathbb{R}$ describing continuum quantum systems with infinitely many degrees of freedom. These can describe systems undergoing a phase change, from rarefied to condensed, at the critical value of a correlation parameter. The method offers promise for generalization to higher spatial dimensions, and for application to theories of extended quantum objects.

This letter describes a new class of quantum models with a countable infinity of degrees of freedom. A parametrized family of such models, regarded as describing a quantized gas of point particles in one-dimensional space, shows a phase transition at the critical value of the correlation parameter κ.

Our construction is based on an approach to quantum theory that has already produced many interesting results. Quantum systems on a spatial manifold $\mathbb{M}$ are described by unitary representations of an infinite-dimensional group $Diff(\mathbb{M})$, the group of diffeomorphisms (i.e. smooth, invertible mappings) of $\mathbb{M}$ under composition [1]. This method, derived from local current algebra, led to a mathematically rigorous prediction of intermediate particle statistics (for $\mathbb{M} = \mathbb{R}^2$) by Goldin *et al*, as had been conjectured by Leinaas and Myrheim; such particles were later termed 'anyons' by Wilczek. This approach yielded important fundamental physical and mathematical properties of anyons—the shifted angular momentum and energy spectra, the connection with configuration-space topology, the relation to the physics of a charged particle circling a region of magnetic flux, and the role of the braid group in anyon statistics [2]. Quantized vortex configurations in ideal, incompressible fluids were also obtained from representations of groups of (area- and volume-preserving) diffeomorphisms of $\mathbb{R}^2$ and $\mathbb{R}^3$, leading to unexpected physical conclusions. For planar fluids pure point vortices are forbidden quantum-mechanically, but one-dimensional *filaments* of vorticity are allowed; similarly, in $\mathbb{R}^3$ pure filaments are kinematically forbidden, while two-dimensional vortex *surfaces*, e.g. ribbons and tubes, can occur [3].

However, a major gap remains. For *unitary* group representations that actually describe the quantum mechanics of extended systems, one needs measures on the infinite-dimensional configuration spaces that are well-behaved under the action of diffeomorphisms (invariant or quasi-invariant, as explained below). This problem must be solved to obtain, for example, fully-quantized non-relativistic strings. Construction of diffeomorphism-invariant measures has also been a long-standing challenge in the programme for finding a consistent theory of quantized gravity [4]. Very recent advances in this direction have been made by Ashtekar and Lewandowski [5], who describe a faithful, diffeomorphism-invariant measure on a compactification of the space of connections modulo gauge transformations. The compactification leads to a space that is, however, extremely large and abstract. In this letter we propose a new method, based on self-similar random processes, for obtaining measures quasi-invariant under diffeomorphisms. We thus have unitary representations for systems modelled on more tractable configuration spaces. In describing the infinite gas we take a step toward the quantum theory for more general extended objects, such as filaments, loops, or ribbons, and we believe our results to be of interest to the programme for the quantization of gravity.

Our most elementary example describes the quantum mechanics of a strictly confined gas of point particles. In generalizing it we quite naturally find parametrized families of models, where the parameter κ relates to correlations among the particles. The quantum systems that result from the corresponding unitary representations of $Diff(\mathbb{R})$ change at the critical value $\kappa = \kappa_0$ from a phase of locally finite configurations with zero average density of particles, to a condensed phase of configurations with a cluster point. Now classical (non-quantized) statistical-mechanical models with phase changes in one dimension include the Kac–Baker model of hard spheres on a line, interacting by means of an exponentially decaying potential, and a family of Ising models with long-range interactions described by Dyson [6]. At the quantum level Lieb *et al* [7] showed it is not necessary to have long-range interactions for phase changes to occur. But the latter are lattice models, as is usual in quantum statistical mechanics. In contrast our method provides new, fully-quantized infinite point-particle models in the continuum, by directly introducing satisfactory probability measures on an infinite-dimensional quantum configuration space. It is not immediately apparent how the systems discussed here can be derived from an assumed interaction, though our construction provides a good deal of intuition.

We next sketch the framework in a self-contained way [8]. A diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ is simply an invertible mapping, with ϕ and ϕ^{-1} smooth. As in [1] we include only diffeomorphisms such that $\phi(x) \to x$ (rapidly, with all derivatives) when $|x| \to \infty$. Then $G = Diff(\mathbb{R})$ is a topological group, whose group law is given by the composition of diffeomorphisms: $(\phi_1\phi_2)(x) = (\phi_2 \circ \phi_1)(x) = \phi_2(\phi_1(x))$. The identity element is the diffeomorphism $e(x) \equiv x$. In general G acts on a space Δ of (quantum) configurations γ, and in this letter Δ is the set of infinite sequences of points drawn from $\mathbb{R}$. The points in a sequence (x_j) are the particle positions in a particular configuration of a gaseous system. For $\gamma = (x_j) \in \Delta$, the transformed configuration $\phi\gamma = (y_j)$ is just the sequence $y_j = \phi(x_j)$, and $(\phi_2 \circ \phi_1)\gamma = \phi_2(\phi_1\gamma)$.

The construction of unitary representations of G requires measures on quantum configuration spaces that are either invariant or quasi-invariant for diffeomorphisms. A measure μ on Δ is called *invariant* for G if for any measurable set $A \subset \Delta$ and $\phi \in G$, $\mu(\phi(A)) = \mu(A)$. The weaker condition of *quasi-invariance* of μ for G states that if A has positive measure, its image $\phi(A)$ has positive measure for any $\phi \in G$, allowing in general $\mu(A) \neq \mu(\phi(A))$. Defining the transformed measure μ_ϕ by $\mu_\phi(A) = \mu(\phi(A))$, quasi-invariance is necessary and sufficient for the existence of the Radon–Nikodym (RN) derivative $(\mathrm{d}\mu_\phi/\mathrm{d}\mu)(\gamma)$. A quasi-invariant measure μ on a configuration space Δ then defines a class of unitary representations V of G in the Hilbert space $\mathcal{H} = L^2_\mu(\Delta, \mathcal{W})$ of square-integrable functions $\Psi(\gamma)$ taking values in a space $\mathcal{W}$, given by

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu}(\gamma)} \tag{1}$$

where $\chi_\phi : \mathcal{W} \to \mathcal{W}$, and the wave function components in $\mathcal{W}$ may describe internal degrees of freedom. Here χ_ϕ satisfies, for all $\phi_1, \phi_2 \in G$,

$$\chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) = \chi_{\phi_1\phi_2}(\gamma) \tag{2}$$

almost everywhere in Δ. Under appropriate technical conditions $V(\phi)$ defines a continuous unitary representation of G in $\mathcal{H}$ whose self-adjoint generators represent the well known Lie algebra $Vect(\mathbb{R})$ of vector fields. These self-adjoint generators, interpreted as momentum density operators, give a *direct physical interpretation* to the representation. The choice $\chi_\phi(\gamma) \equiv 1$ is always permitted, while alternate choices of χ_ϕ are associated with non-trivial phase effects and particle statistics. It is the quasi-invariance of μ that allows the square root of the derivative to occur as a factor in (1), which makes the representation unitary. The construction of such measures when the configuration spaces are (as in our case) infinite dimensional is an in-general unsolved problem, though they are needed in the quantum theory.

For measures on Δ as they are usually constructed in the case of sequence spaces, the RN derivative (if it exists) assumes the form of an infinite product:

$$\frac{\mathrm{d}\mu_\phi}{\mathrm{d}\mu}(\gamma) = \prod_{j=1}^{\infty} u_j\,. \tag{3}$$

Then the issue of quasi-invariance becomes that of the convergence of (3) to a non-zero, non-infinite limit.

One situation that can be understood this way is known to describe the free Bose gas at zero temperature, confirming what we have said thus far [9]. Consider N identical particles uniformly distributed on an interval of length L. The probability that exactly n particles are in a subinterval (a, b) is given by

$$p_N(n; (a, b)) = \frac{N!(b-a)^n(L-b+a)^{N-n}}{n!(N-n)!L^N}\,. \tag{4}$$

Taking the limit as $N, L \to \infty$, with $N/L \to \bar{\rho}$, gives $p(n; (a, b)) = (1/n!)\bar{\rho}^n(b-a)^n \exp[-\bar{\rho}(b-a)]$, i.e. the Poisson distribution. From Kolmogorov's theorem, there exists a unique Borel measure μ on the space Δ of (unordered) locally finite configurations in $\mathbb{R}$, having fixed average density $\bar{\rho}$. The measure μ is called the Poisson measure, with parameter $\bar{\rho}$. The quasi-invariance of μ under diffeomorphisms, shown by other means in [9], can be seen from the fact that $u_j = \mathcal{J}_\phi(x_j)$, the Jacobian of ϕ at x_j. Since ϕ becomes rapidly trivial at infinity, the finiteness of $\bar{\rho}$ implies that (with probability 1) $u_j \to 1$ rapidly as $j \to \infty$. This ensures (3) is positive and finite. The corresponding unitary representations of G describe the free Bose gas.

The measures we now consider are different, in that we permit points to accumulate in a bounded region. Quasi-invariance is then a much more delicate question; for example, permitting the positions of the particles to distribute non-identically but independently does not lead to convergence of the product in (3). Instead we shall let the probability distribution of the position of the jth particle *scale* according to the *outcomes* for the particles previously chosen. This establishes a self-similar random process, that mimics an interacting gas of particles.

For each j, we thus let μ_j be a probability measure on $\mathbb{R}$ contingent on the values of $(x_1, \ldots, x_{j-1})$, with $\mathrm{d}\mu_j(x_j) = f_j(x_j|x_1, \ldots, x_{j-1})\,\mathrm{d}x_j$; where f_j is a probability density function on x_j. The joint probability measure for the first k particles is then $\mu^k = \prod_{j=1}^k \mu_j$. In this way we obtain a compatible family of probability measures (μ^k). By Kolmogorov's theorem, these define a unique measure μ on Δ [10]. The u_j in (3) are given by

$$u_j = \frac{\mathrm{d}\mu_{j\phi}}{\mathrm{d}\mu_j} = \frac{f(\phi(x_j)|\phi(x_1), \ldots, \phi(x_{j-1}))}{f(x_j|x_1, \ldots, x_{j-1})} \mathcal{J}_\phi(x_j) \tag{5}$$

and the quasi-invariance of μ for diffeomorphisms will follow if we choose the f_j so that (3) converges (almost everywhere) to a positive, finite limit.

Next we construct a quasi-invariant measure where (with probability 1) the particles accumulate. Choose the first pair of particle positions (x_1, x_2) from any non-vanishing probability density $f_1(x) = f_2(x)$. Choose the second pair of particle positions from the *uniform* density on the interval $[x_1, x_2]$; i.e.

$$f_3(x|x_1, x_2) = f_4(x|x_1, x_2) = \frac{\chi_{[x_1,x_2]}(x)}{|x_2 - x_1|} \tag{6}$$

where $\chi_{[a,b]}$ denotes the characteristic function (indicator function) of the interval $[a, b]$. Iterating this process, choose (x_{2m+1}, x_{2m+2}) from the uniform density on $[x_{2m-1}, x_{2m}]$. We have, in fact, a Markov process. It is easy to show that the measure μ is concentrated on convergent sequences, whence

$$u_{2m+1} = \frac{|x_{2m} - x_{2m-1}|}{|\phi(x_{2m}) - \phi(x_{2m-1})|} \mathcal{J}_\phi(x_{2m+1}) \to 1 \tag{7}$$

with probability 1, and similarly for u_{2m+2}. In the structure of expression (7) for u_j we can see clearly how the conditional probability enters: the first factor approaches the reciprocal of the Jacobian as the x_j approach their limit. The way in which the width of the distribution for each pair of points is determined directly by the outcome of choosing the previous pair builds a kind of scale invariance or self-similarity into the configurations, which is *just what is needed* for quasi-invariance. Of course, equation (7) is a necessary condition for the convergence of the infinite product (3), but it is not sufficient. The quasi-invariance of μ follows because our construction also ensures that the *rate* of convergence is sufficiently rapid. We have, by elementary methods, the sufficient condition

$$\sum_{j=1}^{\infty} |x_{j+1} - x_j| < \infty \tag{8}$$

(with probability 1). The quasi-invariance of μ means that a unitary representation of the diffeomorphism group, and thus a consistent quantum mechanics, exists for these configurations! Physically, this elementary model may be interpreted as describing a strictly confined 'cluster' formed from infinitely many particles.

It is interesting that the infinite free Bose gas in one dimension (i.e. Poisson measure with parameter $\bar{\rho}$), can be obtained straightforwardly in this framework as a kind of 'reciprocal' of the above Markov process. One chooses the positions of successive particles to be *outside* the intervals established by preceding choices, with densities f_j decaying exponentially in both directions, and with $\bar{\rho}$ independent of j in the exponential distributions.

Now the previous example depends critically on the fact that orientation-preserving diffeomorphisms of $\mathbb{R}$ respect 'betweenness'—so that regions of positive measure (with the points x_{2m+1} and x_{2m+2} falling as they must between x_{2m-1} and x_{2m}) cannot be mapped by $\phi \in G$ into regions of zero measure (with x_{2m+1} or x_{2m+2} outside the interval). Next we remove our reliance on this feature, in anticipation of future generalization to higher dimensions. To do this, we must use successive probability densities that are *nowhere* vanishing.

It might be thought that one could proceed by choosing a point x_0 from a non-vanishing density, fixing a standard deviation σ_0, and then choosing the x_j from normal distributions with mean x_0 and shrinking standard deviations $\sigma_j = 2^{-j}\sigma_0$. Such a process describes points on a Brownian path, obtained from a Wiener measure. However, though x_j converges (with probability 1) to x_0, the measure is *not* quasi-invariant for diffeomorphisms. The convergence of x_j and the condition on σ_j (independent of the outcomes $x_1, \ldots, x_{j-1}$) are not sufficient to control the behaviour either of the ratio $[f(\phi(x_j)|\phi(x_1), \ldots, \phi(x_{j-1}))/f(x_j|x_1, \ldots, x_{-1})]$ or of the Jacobian $\mathcal{J}_\phi(x_j)$. Brownian paths in $\mathbb{R}^n$ have (with probability 1) fixed second variation according to the covariance matrix of the Brownian motion, while diffeomorphisms of $\mathbb{R}^n$ act so as to change the covariance matrix. Thus we cannot use Wiener measure to achieve the goal. The idea that allows a breakthrough is again the *self-similarity* of the random process.

To illustrate the role played by self-similarity consider the following model. Again choose the first two points x_0 and x_1 from non-vanishing densities f_0 and f_1 respectively. Having chosen the points $x_0, \ldots, x_m$, choose x_{m+1} from a normal distribution; let the mean for this normal be x_0, and let the standard deviation be $\sigma_m = \kappa\,|x_m - x_0|$. Here $\kappa > 0$ is a *correlation parameter* independent of m, and small values of κ correspond to more tightly bound systems. Thus

$$f^{\kappa}_{m+1}(x_{m+1}|x_m, x_0) = \frac{(2\pi)^{-\frac{1}{2}}}{\kappa|x_m - x_0|}\exp\left[-\frac{1}{2\kappa^2}\left(\frac{x_{m+1} - x_0}{x_m - x_0}\right)^2\right]. \tag{9}$$

We can now show that for small values of κ the sequence (x_j) converges to a finite limit (with probability 1), and that $u_j \to 1$ sufficiently rapidly to ensure convergence of the infinite product (3), where the terms u_j in (5) have been defined from (9). Furthermore, we can show that the system undergoes a phase transition at a critical value κ_0 of the parameter κ from the 'rarified' to the 'condensed' phase. More precisely, we can prove there exists a κ_0 such that if $\kappa < \kappa_0$ sequences converge with probability 1, while if $\kappa > \kappa_0$, sequences diverge with probability 1; the associated measures on Δ are quasi-invariant for diffeomorphisms.

The idea is as follows. From the self-similarity of the random process whereby successive densities are constructed, we have that the probability distribution for $y_m = (x_m - x_0)/|x_{m-1} - x_0|$, i.e. the distribution of $(x_m - x_0)$ in standard deviation units, is a fixed normal distribution (depending only on κ), independent of m. Consider the random variable $(\log|x_m - x_0|)$. Since the distributions of the y_m do not depend on m, we can identify $(\log|x_m - x_0|)$ with a random walk on the real line. The step of this random walk is $\xi = \log|y_m| = \log|x_m - x_0| - \log|x_{m-1} - x_0|$, and the probability density for ξ is given by

$$f(\xi) = \frac{2(2\pi)^{-\frac{1}{2}}}{\kappa}\exp\xi\exp\left(-\frac{1}{2\kappa^2}\exp 2\xi\right). \tag{10}$$

A standard application of the central limit theorem [11] then implies that if $E\xi = \int \xi f(\xi) < 0$, the random walk $(\log|x_m - x_0|)$ drifts to $-\infty$ (with probability 1), which is equivalent to the statement that (x_m) converges to x_0 (with probability 1). On the other side, if $E\xi > 0$, then $(\log|x_m - x_0|)$ drifts to $+\infty$ and $|x_m| \to \infty$ (with probability 1). The critical value of

κ occurs when

$$\int_{-\infty}^{\infty} \xi \exp\xi \exp\left(-\frac{1}{2\kappa^2}\exp 2\xi\right) \mathrm{d}\xi = \frac{\kappa\sqrt{2\pi}}{4}\left(\log\frac{\kappa^2}{2} - \gamma\right) = 0 \tag{11}$$

i.e. $\kappa_0^2/2 = \exp\gamma$, where $\gamma = 0.5772\ldots$ is the Euler–Mascheroni constant; the definite integral in (11) was evaluated using [12, equation 4.355.1]. Proving the actual quasi-invariance of the measures associated with these Markov processes (needed for the existence of the corresponding quantum models) is somewhat more complicated, requiring that a condition like (8) be demonstrated. We note that nothing we have done actually depends on the use of normal distributions. All that is really necessary is the scaling property and the applicability of the central limit theorem.

In the above we have treated the particles as at the outset distinguishable, in that measures are constructed on *ordered* sequences (x_j). However, the physics does not depend on the labelling of points that derives from their positions in the sequences. In one space dimension it is straightforward to 'sum over permutations' by reconstructing the measures in terms of a physical labelling in which, for any configuration, points are indexed according to the positions they actually assume on the real line.

Thus for a whole class of models there is a critical value of κ. For sufficiently large κ the sequence (x_j) diverges with probability 1, and has zero average density (rarefied phase); for sufficiently small κ it converges with probability 1, and the particles accumulate (condensed phase). We believe that other non-trivial phenomena in quantum statistical mechanics can also be modelled by continuous unitary representations of the group G.

Finally we conjecture that a procedure similar to that given by (9) will work in n space dimensions, $n > 1$, to give measures quasi-invariant for $Diff(\mathbb{R}^n)$. Here it will be necessary to choose successive points x_{m+1} based on the outcomes for the preceding n choices $(x_{m-n+1}, \ldots, x_m)$, using these outcomes to define the covariance matrix of a multivariate normal distribution.

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